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Long waves in a uniform channel of arbitrary cross-section

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Equations of motion are derived for long gravity waves in a straight uniform channel. The cross-section of the channel may be of any shape provided that it does not have gently sloping banks and it is not very wide compared with its depth. The equations may be reduced to those for two-dimensional motion such as occurs in a rectangular channel. The order of approximation in these equations is sufficient to give the solitary wave as a solution.

1. Introduction

The waves considered are irrotational gravity waves on the surface of water, travelling along a straight channel of uniform cross-section. The cross-section of the channel is assumed to have its breadth and depth of comparable size and the slope of its banks is assumed to be $O(1)$ or vertical. That is, very wide channels and channels with very gently sloping banks are excluded. Typical examples are channels with semi-circular or trapezoidal cross-sections.

Scott Russell (1844) performed experiments with solitary waves in channels of various non-rectangular shapes, mostly triangular. He found that the wave was higher and shorter where the water was shallower. In a channel of the form of a right-angled isosceles triangle, with the hypotenuse horizontal, the solitary waves maintained their unity of form. However, in a triangular channel with each side having a slope of one in four, although a single wave propagated, it broke at the edges. In a very broad channel the wave did not have a coherent form, but its various parts moved with a velocity appropriate to the local depth of water.

The equation of motion for infinitesimal long waves in such channels had been given by Kelland (1839, see Lamb 1932, §169). Particular solutions for 'short' waves with infinitesimal amplitude in various triangular channels are summarized by Lamb (1932, §261). Peters (1966) has given the theory of the solitary wave in channels of arbitrary cross-section and has also included the effects of an initial vorticity distribution. Here, only irrotational flows are treated, but the equations of motion for long waves such as the solitary wave are derived and it is shown how they may be transformed to the corresponding equations for two-dimensional motion in many cases, thus making the solutions of those equations applicable to more general channels.

If a long wave of small amplitude travels along a channel, the motion of the water is almost entirely along the channel, the surface elevation is nearly uniform across the channel and the pressure is practically hydrostatic. The corresponding equation describing the flow is the one-dimensional wave equation, which has solutions which indicate that any wave may travel unchanged with constant velocity. However, this is often inadequate to describe the behaviour of quite small waves. For rectangular channels the appropriate next approximation has been known for a considerable time (Boussinesq 1871). It is necessary to include both the second-order effects of the amplitude and the effect of the vertical acceleration of the water on the pressure in order to get a uniformly valid approximation. This approximation may be characterized by the solitary wave which is the only isolated wave to travel along a channel without change of form. Periodic solutions representing a train of waves were found by Korteweg & de Vries (1895), who also found a simpler form of the equations of motion when waves travel in one direction only. These are the only analytical solutions but numerical methods have been used to obtain unsteady solutions (e.g. Long 1964; Peregrine 1966).

There are two ways to approach three-dimensional problems in shallow-water waves. One is to suppose that motions in all horizontal directions are of the same magnitude, with much smaller vertical velocities. This method has been used to obtain equations for waves on water of variable depth (Peregrine 1967). The other approach, which is used here, is to assume that velocities in one direction, along the channel in this case, are much larger than vertical or transverse velocities. This implies that the width of the channel may not be very much greater than its depth. In the last section the case of a wide trapezoidal channel is discussed and it is shown that the approximation breaks down for very wide channels.

2. Equations of motion: preliminaries

Cartesian axes $Oxyz$ are introduced with the origin in the undisturbed free surface, Ox directed along the channel, Oy across the channel and Oz vertical, as indicated in figure 1. The water in the channel is taken to be inviscid and incompressible, with no surface tension. The motion of the water is assumed to be irrotational initially so that it remains irrotational in the absence of breaking waves.

The density ρ of the water, the acceleration g due to gravity, and a typical undisturbed depth h_0 of water, are used to introduce dimensionless variables as follows:

$$(x, y, z) = (x^*, y^*, z^*)/h_0, \quad t = t^*(g/h_0)^{\frac{1}{2}},$$
$$(u, v, w) = \mathbf{u} = \mathbf{u}^*(gh_0)^{-\frac{1}{2}}, \quad p = (p^* - p_0)/\rho g h_0,$$

where * indicates a dimensional variable and p_0 is the atmospheric pressure above the water. The equations of motion in these variables are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\,.\,\nabla)\mathbf{u} + \nabla p + (0, 0, 1) = 0. \tag{1}$$

The continuity equation is

$$\nabla\,.\,\mathbf{u} = 0, \tag{2}$$

which may also be written in an integrated form,

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0, \tag{3}$$

where $A(x,t)$ is the cross-sectional area of the channel filled with water and

$$Q(x,t)=\iint_A u\,dy\,dz \tag{4}$$

is the total flow along the channel at any instant. The irrotational flow condition is

$$\nabla\times\mathbf{u}=0. \tag{5}$$

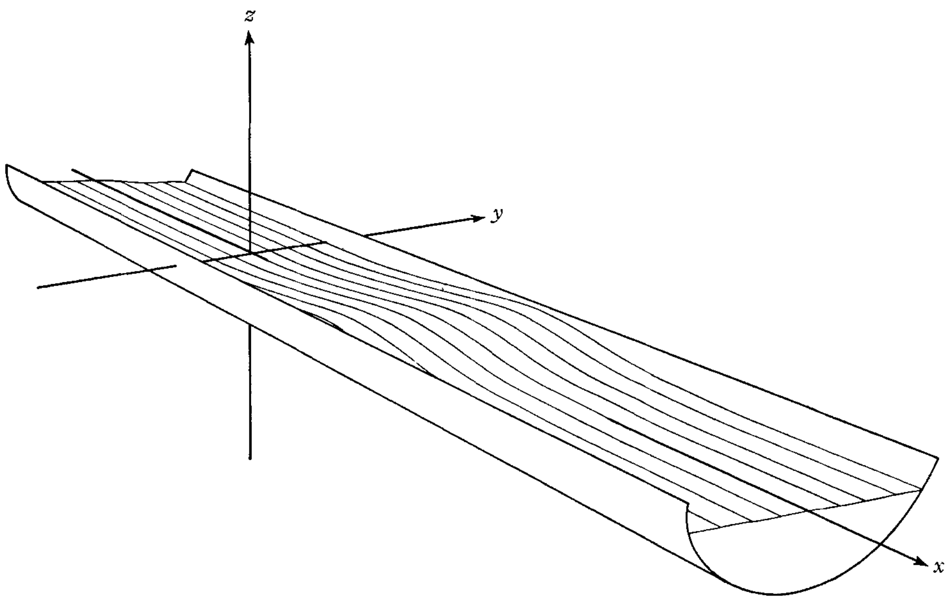

FIGURE 1. Arrangement of co-ordinate axes.

The free surface is taken to be $z=\zeta(x,y,t)$ and the boundary conditions there are

$$p=0 \quad\text{and}\quad \frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}=w.$$

The boundary condition on the channel is that the normal velocity is zero.

The above equations formulate the mathematical problem, but to make further progress it is necessary to make approximations which involve the properties of the waves under consideration. For shallow-water waves there are two relevant non-dimensional parameters. One, σ, = (depth of water)/(wavelength), so that a typical scale of variation with x is σ^{-1}, and, since long waves are being considered, $\sigma \ll 1$. The other parameter, ϵ, is a measure of the amplitude of waves compared with the depth of water. That is, $\zeta = O(\epsilon)$.

If $\epsilon = O(1)$, approximate equations of motion may be found, which are exactly the same as for a rectangular channel with the same mean depth. But these finite-amplitude equations are not uniformly valid since they indicate that the forward-facing slopes of waves grow steeper (see Stoker 1957), so that eventually the approximation $\sigma \ll 1$ no longer holds. For solitary wave theories it is necessary to assume $\epsilon \ll 1$ and $\epsilon \sim \sigma^2$ (Ursell 1953), so σ will be put equal to $\epsilon^{\frac{1}{2}}$ in what follows.

These assumptions are sufficient to fix the order of magnitude of all the other variables. All the dependent variables are now expanded as power series in ϵ, and the independent variables x and t are also scaled appropriately,

$$x_1 = \epsilon^{\frac{1}{2}}x \quad \text{and} \quad t_1 = \epsilon^{\frac{1}{2}}t.$$

This has two aims, one to show explicitly the order of magnitude of each term, and the other to provide a systematic basis for finding higher approximations.

Two variables, the pressure and cross-sectional area, are not small in the wave since they do not vanish when the water is undisturbed. Thus p and A are expanded in the form

$$f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots.$$

The variables ζ, u, Q are all $O(\epsilon)$ and thus are expanded

$$\epsilon f_1 + \epsilon^2 f_2 + \dots.$$

It is possible to have an initial flow u_0, Q_0 as in Peters's (1966) approach. The velocities v and w are $O(\epsilon^{\frac{3}{2}})$ and are expanded

$$\epsilon^{\frac{1}{2}}(\epsilon f_1 + \epsilon^2 f_2 + \dots).$$

When these new scaled variables have been substituted into the equations and boundary conditions it is possible to group terms of the same order of magnitude together. They are then separately put equal to zero in order to find successively the terms in the expansion of the variables.

Since the undisturbed condition is taken to be still water, the only equations $O(1)$ are

$$\frac{\partial p_0}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p_0}{\partial z} + 1 = 0,$$

with the boundary condition $p_0 = 0$ at $z = 0$. The solution is the hydrostatic one, $p_0 = -z$. The equation of order $\epsilon^{\frac{1}{2}}$ from (3), $\partial A_0/\partial t = 0$, shows A_0 to be a function of x only. It is a constant for a uniform channel.

3. Equations of motion: first approximation

It is now convenient to introduce a two-dimensional vector operator, $\nabla_1 = (0, \partial/\partial y, \partial/\partial z)$, and a two-dimensional velocity potential $\phi_1(x_1, y, z, t_1)$ such that

$$v_1 = \frac{\partial \phi_1}{\partial y} \quad \text{and} \quad w_1 = \frac{\partial \phi_1}{\partial z}.$$

This is permissible since the equation $O(\epsilon^{\frac{3}{2}})$ in the irrotationality condition (5) is

$$\frac{\partial v_1}{\partial z} = \frac{\partial w_1}{\partial y}.$$

The other two equations in (5) give, to $O(\epsilon)$,

$$\nabla_1 u_1 = 0,$$

and hence

$$u_1 = u_1(x_1, t_1).$$

The $O(\epsilon)$ terms in the y and z equations of motion give only

$$\nabla_1 p_1 = 0.$$

The boundary condition for p at the free surface to $O(\epsilon)$ is that

$$p_0 + \epsilon p_1 = 0 \quad \text{at} \quad z = \epsilon\zeta_1.$$

Therefore $p_1 = p_1(x_1, t_1) = \zeta_1(x_1, t_1)$, and ζ_1 is independent of y.

The x equation of motion has terms of $O(\epsilon^{\frac{3}{2}})$,

$$\frac{\partial u_1}{\partial t_1} + \frac{\partial p_1}{\partial x_1} = 0;$$

which may be rewritten

$$\frac{\partial u_1}{\partial t_1} + \frac{\partial \zeta_1}{\partial x_1} = 0. \tag{6}$$

To the same order the continuity equation (3) is

$$\frac{\partial A_1}{\partial t_1} + \frac{\partial Q_1}{\partial x_1} = 0.$$

However, if $B(z)$ is the breadth of the channel at height z,

$$A = A_0 + \int_0^\zeta B(z)\,dz$$
$$= A_0 + \epsilon\zeta_1 B(0) + O(\epsilon^2),$$

so that $A_1 = B_0\zeta_1$, where $B_0 = B(0)$. From the definition of Q, $Q_1 = A_0 u_1$, so that the continuity equation becomes

$$B_0 \frac{\partial \zeta_1}{\partial t_1} + A_0 \frac{\partial u_1}{\partial x_1} = 0. \tag{7}$$

The elimination of u_1 from (6) and (7) gives

$$\frac{\partial^2 \zeta_1}{\partial t_1^2} = \frac{A_0}{B_0}\frac{\partial^2 \zeta_1}{\partial x_1^2};$$

that is, the wave equation with a wave velocity, c_0, such that

$$c_0^2 = A_0/B_0 = \text{mean depth of channel}.$$

The transverse velocity potential ϕ_1 may now be found from the $O(\epsilon^{\frac{3}{2}})$ terms of the continuity equation (2), and the kinematic boundary conditions. The equation is

$$\frac{\partial u_1}{\partial x_1} + \nabla_1^2 \phi_1 = 0,$$

and the boundary conditions are $\partial\phi_1/\partial n = 0$ on solid boundaries, and

$$\frac{\partial \zeta_1}{\partial t_1} = \frac{\partial \phi_1}{\partial z} \quad \text{on} \quad z = 0,$$

to this approximation. This last boundary condition may be rewritten as

$$\frac{\partial \phi_1}{\partial z} = -c_0^2 \frac{\partial u_1}{\partial x_1}$$

by using (7). This suggests the assumption

$$\phi(x_1, y, z, t_1) = -\frac{\partial u_1}{\partial x_1}\psi(y, z),$$

where $\psi(y, z)$ satisfies

$$\nabla_1^2\psi = 1, \quad \text{with} \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on solid boundaries}$$

and
$$\frac{\partial \psi}{\partial z} = c_0^2 \quad \text{on} \quad z = 0. \tag{8}$$

This is a well-posed Neumann problem for ψ; its solution for ψ exists and is unique within an arbitrary constant. Thus the solution for ϕ_1 is

$$-\frac{\partial u_1}{\partial x_1}\psi + \chi(x_1, t_1), \tag{9}$$

where χ is arbitrary and is not of interest in this approximation since it does not affect v_1 or w_1.

4. Equations of motion: second approximation

It is at the second approximation that variations of ζ and u across the channel appear. In two-dimensional flow the vertical acceleration of the water becomes significant: here the transverse acceleration is as important. Their effects appear through the terms of $O(\epsilon^2)$ in the y and z equations of motion which give

$$\frac{\partial}{\partial t}\nabla_1\phi_1 + \nabla_1 p_2 = 0.$$

This equation integrates to

$$p_2 = \frac{\partial^2 u_1}{\partial x_1 \partial t_1}\psi(y, z) + H(x_1, t_1),$$

where H is an arbitrary function related to ζ_2 by the boundary condition on the pressure which reduces to

$$-\zeta_2 + p_2 = 0 \quad \text{on} \quad z = 0.$$

That is
$$\zeta_2 = \frac{\partial^2 u_1}{\partial x_1 \partial t_1}\psi(y, 0) + H(x_1, t_1). \tag{10}$$

This equation shows that the transverse variation of ζ_2 is like $\psi(y, 0)$. $H(x_1, t_1)$ incorporates the function $\chi(x_1, t_1)$ in (9).

The $O(\epsilon^2)$ terms in the y and z components of (5) give

$$\nabla_1 u_2 = \frac{\partial}{\partial x_1}\nabla_1\phi_1,$$

so that
$$u_2 = -\frac{\partial^2 u_1}{\partial x_1^2}\psi(y, z) + U(x_1, t_1),$$

where U is another arbitrary function.

The $O(\epsilon^2)$ terms in the x equation of motion are

$$\frac{\partial u_2}{\partial t_1}+u_1\frac{\partial u_1}{\partial x_1}+\frac{\partial p_2}{\partial x_1}=0,$$

and thus become

$$\frac{\partial U}{\partial t_1}+u_1\frac{\partial u_1}{\partial x_1}+\frac{\partial H}{\partial x_1}=0. \tag{11}$$

The next approximation to the cross-sectional area is a little more complicated:

$$\begin{aligned} A &= A_0+\int_0^{\epsilon\zeta_1+\epsilon^2 H}[B_0+zB'(0)+O(\epsilon^2)]\,dz \\ &\qquad +\epsilon^2\frac{\partial^2 u_1}{\partial x_1\partial t_1}\int_{B_0}\psi(y,0)\,dy+O(\epsilon^3) \\ &= A_0+\epsilon B_0\zeta_1+\epsilon^2\left(\tfrac{1}{2}B_1\zeta_1^2+B_0H+B_0\psi_B\frac{\partial^2 u_1}{\partial x_1\partial t_1}\right)+O(\epsilon^3), \end{aligned} \tag{12}$$

where $B_1 = B'(0)$, and

$$\psi_B=\frac{1}{B_0}\int_{B_0}\psi(y,0)\,dy. \tag{13}$$

If the waterline of a channel is very gently sloping, $B'(0)$ is large and the term $B_1\zeta^2$ may no longer be considered as a second-order term. This theory is therefore only applicable to channels for which $B'(0)$ is $O(1)$.

The calculation for Q is similar to that for A and yields

$$Q=\epsilon A_0u_1+\epsilon^2\left(B_0u_1\zeta_1+A_0U-A_0\psi_A\frac{\partial^2 u_1}{\partial x_1^2}\right)+O(\epsilon^3), \tag{14}$$

where

$$\psi_A=\frac{1}{A_0}\iint_{A_0}\psi(y,z)\,dy\,dz. \tag{15}$$

Expressions (12) and (14) give A_2 and Q_2, which may then be substituted into the $O(\epsilon^{\frac{5}{2}})$ terms from equation (3) to give a second equation for U and H.

However, while this approach is reasonably straightforward for finding approximations to the equations of motion, it is not so convenient for finding solutions of the equations. For example, if u and ζ are given as functions of x at $t = 0$, we can put u_1 and ζ_1 equal to the corresponding scaled functions at $t_1 = 0$ and then solve equations (6) and (7) to find u_1 and ζ_1 at later times. These can then be used in the second-order equations to give U and H and hence y_2 and ζ_2. In general, however, such a procedure leads to functions U and H varying directly with t_1 and hence to solutions which are likely to be valid only for $t_1 = O(1)$. A more profitable approach for finding solutions is to use equations which are correct to second order initially; then, this problem does not appear to occur. Although in the two-dimensional case the only known analytical solutions are steady translational waves, numerical solutions give reasonable results with unsteady motions for relatively large times (Peregrine 1966).

The first- and second-order equations can be combined by using variables correct to the second order. For example, for the height of the water surface,

$$\eta(x,t)=\epsilon\zeta_1(x_1,t_1)+\epsilon^2 H(x_1,t_1) \tag{16}$$

is an obvious choice. The choice is not so clear for a velocity variable. One can use

$$u'(x,t) = \epsilon u_1(x_1,t_1) + \epsilon^2 U(x_1,t_1), \tag{17}$$

or

$$\bar{u}(x,t) = Q/A = \epsilon u_1 + \epsilon^2 \left(U - \psi_A \frac{\partial^2 u_1}{\partial x_1^2} \right). \tag{18}$$

The resulting equations differ in the second-order terms, but these terms can be changed by using the first-order relationships and the equations are equivalent. (For example, in equation (18) $\partial^2 u_1/\partial x_1^2$ may be replaced by $(1/c_0^2)\,\partial^2 u_1/\partial t_1^2$.)

The momentum equation is obtained by adding ϵ times equation (6) to ϵ^2 times equation (11). The dependent variables are changed to η and u', and at the same time x_1 and t_1 are replaced by x and t in order to remove all ϵ, giving

$$\frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x} + \frac{\partial \eta}{\partial x} = 0. \tag{19}$$

Similarly from the continuity equation, after dividing by B_0,

$$\frac{\partial}{\partial t}\left(\eta + \tfrac{1}{2}b\eta^2 + \psi_B \frac{\partial^2 u'}{\partial x\,\partial t}\right) + \frac{\partial}{\partial x}\left(c_0^2 u' + \eta u' - c_0^2 \psi_A \frac{\partial^2 u'}{\partial x^2}\right) = 0, \tag{20}$$

where $b = B_1/B_0 = B'(0)/B(0)$. This equation may be rewritten

$$\frac{\partial}{\partial t}(\eta + \tfrac{1}{2}b\eta^2) + \frac{\partial}{\partial x}(c_0^2 u' + \eta u') + (\psi_B - \psi_A)\frac{\partial^3 u'}{\partial x\,\partial t^2} = 0, \tag{21}$$

where only the difference $\psi_B - \psi_A$, which is unique, appears, instead of ψ_A or ψ_B, which may have arbitrary constants added.

5. Transformation to the equations for a rectangular channel

For a rectangular channel of non-dimensional depth h, $b = 0$, $c_0^2 = h$ and $\psi = \frac{1}{2}z^2 + hz$, so that $\psi_B - \psi_A = \frac{1}{3}h^2$. Thus for any channel with sides which are vertical at the waterline, so that $b = 0$, the equations (19) and (21) can be transformed to those for a rectangular channel of the same mean depth by introducing

$$x' = [3(\psi_B - \psi_A)]^{-\frac{1}{2}} c_0^2 x$$

and

$$t' = [3(\psi_B - \psi_A)]^{-\frac{1}{2}} c_0^3 t.$$

If $b \neq 0$ a transformation is still possible if attention is restricted to waves travelling in one direction only. In this case, corresponding to the simplification of Korteweg & de Vries (1895), the equations (19) and (21) may be reduced to

$$2\frac{\partial u'}{\partial t} + 2c_0\frac{\partial u'}{\partial x} + (3 - bc_0^2)\,u'\frac{\partial u'}{\partial x} = (\psi_B - \psi_A)\frac{\partial^3 u'}{\partial x^2\,\partial t}, \tag{22}$$

with

$$\eta = c_0 u' + O(\epsilon^2).$$

This equation is derived in an appendix. The transformation to the corresponding form for a rectangular channel of the same mean depth involves only the above change to x' and t' and a new velocity variable,

$$u'' = (1 - \tfrac{1}{3}bc_0^2)\,u'. \tag{23}$$

6. Discussion

The transformation from x and t to x' and t' is only a slight change in the time and length scales, and since they are both changed by the same amount the velocity of waves in a channel with $b = 0$ is the same as in a rectangular channel. For example, a solitary wave of amplitude a^* in a rectangular channel of depth h^* has a velocity $(gh^*)^{\frac{1}{2}}(1+a^*/2h^*)$, which in non-dimensional terms is $c_0(1+a/2c_0^2)$. This expression is thus the velocity of a solitary wave in any channel for which $b = 0$.

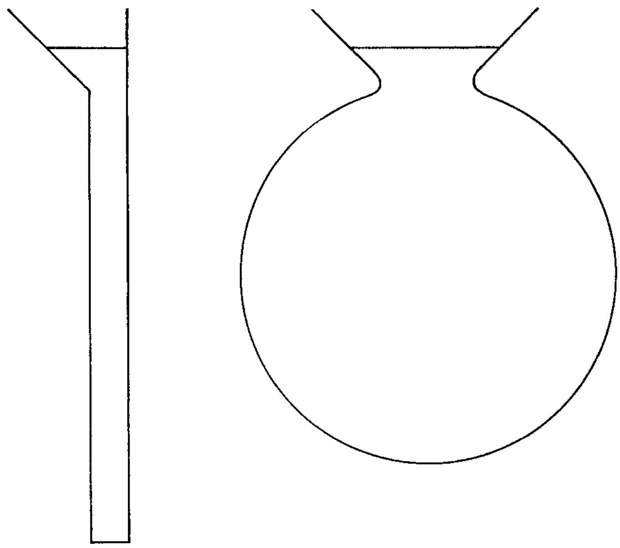

FIGURE 2. Cross-sections with $bc_0^2 > 3$.

On the other hand, if $b \neq 0$, the transformation (23) implies a velocity

$$c_0[1+(1-\tfrac{1}{3}bc_0^2)\,a/2c_0^2] \tag{24}$$

for a solitary wave with non-dimensional amplitude a. It is of interest to note that this velocity does not depend on the function ψ but solely on the geometry of the channel. Although it is presented in a different form this is the same as the result found by Peters (1966). It is not necessary to use these transformations since the solitary wave solution may be found directly from (22) or (19) and (21). It is

$$\eta = a\,\mathrm{sech}^2 \frac{1}{2}\left\{\frac{a(1-\frac{1}{3}bc_0^2)}{c_0^2(\psi_B-\psi_A)}\right\}^{\frac{1}{2}}(x-ct).$$

It will be noticed that, if $bc_0^2 = 3$, equation (22) is linear and there is no solitary wave solution; while if $bc_0^2 > 3$ then the solitary wave is below the mean surface instead of the usual positive wave. The condition $bc_0^2 > 3$ may be written

$$\frac{B'(0)A_0}{B_0^2} > 3, \tag{25}$$

and is a purely geometrical one. A relatively large area is required relative to the breadth. Two cross-sections satisfying this condition are shown in figure 2.

Although they are rather unusual it seems possible that a negative solitary wave could be generated in such a channel. However, it might be more difficult to identify than the usual positive wave, since, while the positive wave travels faster than any wave of smaller amplitude, the negative wave would travel slower than smaller long waves, but shorter waves also travel slower than long waves, so the two may be confused.

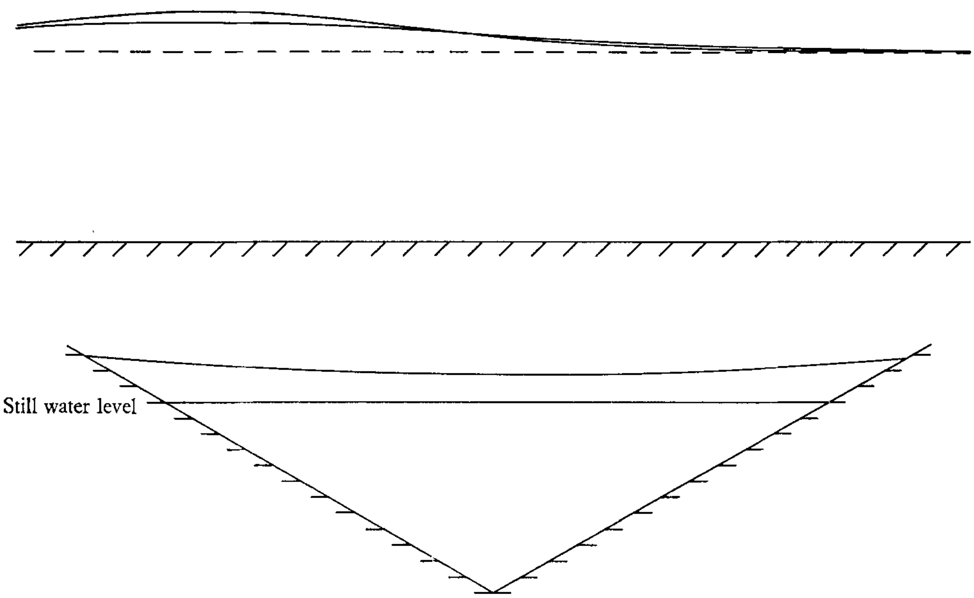

FIGURE 3. Solitary wave in a triangular channel, with the sides of the channel at 60° to the vertical and the amplitude of the wave at the centre equal to one third of the mean depth. (*a*) Side view: the higher and steeper line is the profile at the side of the channel, the other line is the profile at the centre. (*b*) View along the channel.

The transverse velocities and transverse variation of crest height are obtained from the function $\psi(y,z)$. There are are only a few cases for which it may be evaluated explicitly. It has already been given for a rectangle, and Peters (1966) gives an expression for its value in a semi-circle. There is a particularly simple solution for a triangle; if the origin is transferred to the bottom corner of the triangle, the upper edge being the free surface, then

$$\psi = \tfrac{1}{4}(y^2+z^2).$$

It is easy to see that this implies that all motion takes place in planes passing through the bottom corner of the triangle. Figure 3 shows the appearance of a solitary wave in a triangular channel according to this theory.

7. Wide channels

In a very wide channel of rectangular cross-section it is possible to have a solitary wave travelling along the channel with its crest-line perpendicular to the sides of the channel. It is reasonable to suppose that, if one side of the channel is not vertical, but slopes at some angle to the vertical, it may be possible for a solitary wave to propagate as before but with a slightly different shape and velocity distribution near the sloping side. It would also be reasonable to suppose that this theory is applicable. However, it is not. It predicts large variations in crest height due to the sloping side.

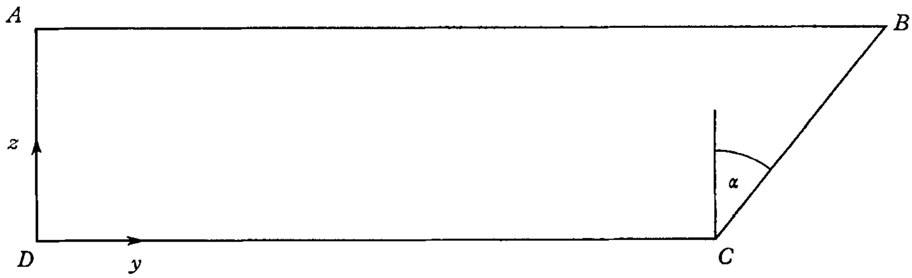

FIGURE 4. Cross-section of a wide channel.

Consider the cross-section in figure 4, where $AD = 1$, $DC = L \gg 1$, and CB is at an angle α to the vertical. The boundary conditions on ψ are that

$$\frac{\partial \psi}{\partial n} = 0 \quad \text{on } ADCB \quad \text{and} \quad \frac{\partial \psi}{\partial z} = \frac{L + \frac{1}{2}\tan\alpha}{L + \tan\alpha} \quad \text{on } AB.$$

The boundary conditions on $BADC$ and the equation $\nabla_1^2 \psi = 1$ are satisfied by

$$\psi_0 = \frac{\frac{1}{4}\tan\alpha}{L + \tan\alpha} y^2 + \frac{L + \frac{1}{2}\tan\alpha}{2(L + \tan\alpha)} z^2,$$

so that, if $\psi = \psi_0 + \psi_1$, then ψ_1 is a harmonic function with $\partial\psi_1/\partial n = 0$ on $BADC$ and

$$\frac{\partial \psi_1}{\partial n} = \frac{(z - \frac{1}{2})L \sin\alpha}{L + \tan\alpha} \quad \text{on } CB.$$

The variation in crest height across the channel is given by $\psi(y, 1)$. For large L, ψ_1 will contribute $O(1)$ near B but will diminish exponentially away from B. ψ_0, however, changes by $O(L)$ between A and B. This indicates a large change in amplitude across the wave, and, since variation across the wave is assumed to be small in the theory, it shows that this theory is not applicable to very wide channels. It is likely that in practice even waves of very small amplitude will break at the shore of wide channels, or channels with gently shelving edges.

Appendix. Derivation of equation (22)

In equations (19) and (21) put

$$c^2 = c_0^2 + \eta,$$

and divide (21) by c. The sum and difference of the resulting equations may be written

$$\left[\frac{\partial}{\partial t} + (u' + c)\frac{\partial}{\partial x}\right](u' + 2c) + 2b(c^2 - c_0^2)\frac{\partial c}{\partial t} + \frac{(\psi_B - \psi_A)}{c}\frac{\partial^3 u'}{\partial x\, \partial t^2} = 0 \qquad \text{(A 1)}$$

and

$$\left[\frac{\partial}{\partial t} + (u' - c)\frac{\partial}{\partial x}\right](u' - 2c) - 2b(c^2 - c_0^2)\frac{\partial c}{\partial t} - \frac{(\psi_B - \psi_A)}{c}\frac{\partial^3 u'}{\partial x\, \partial t^2} = 0. \qquad \text{(A 2)}$$

Now, reintroduce x_1 and t_1 and put

$$c = c_0 + \epsilon c_1 + \epsilon^2 c_2 + \dots$$

and

$$u' = \epsilon u_1 + \epsilon^2 u_2 + \dots$$

in these equations. The first-order terms are

$$\left(\frac{\partial}{\partial t_1} + c_0 \frac{\partial}{\partial x_1}\right)(u_1 + 2c_1) = 0$$

and

$$\left(\frac{\partial}{\partial t_1} - c_0 \frac{\partial}{\partial x_1}\right)(u_1 - 2c_1) = 0.$$

The solution corresponding to waves travelling in the $+x$ direction only is

$$u_1 + 2c_1 = f(x_1 - c_0 t_1)$$

and

$$u_1 - 2c_1 = 0.$$

Hence,

$$\eta = \epsilon c_0 u_1 + O(\epsilon^2).$$

When this solution is used to eliminate c_1, the second-order equations obtained from (A 1) and (A 2) are

$$\left(\frac{\partial}{\partial t_1} + c_0 \frac{\partial}{\partial x_1}\right)(u_2 + 2c_2) + 3u_1 \frac{\partial u_1}{\partial t_1} + bc_0 u_1 \frac{\partial u_1}{\partial t_1} + \frac{(\psi_B - \psi_A)}{c_0} \frac{\partial^3 u_1}{\partial x_1 \partial t_1^2} = 0 \quad \text{(A 3)}$$

and

$$\left(\frac{\partial}{\partial t_1} - c_0 \frac{\partial}{\partial x_1}\right)(u_2 - 2c_2) - bc_0 u_1 \frac{\partial u_1}{\partial t_1} - \frac{(\psi_B - \psi_A)}{c_0} \frac{\partial^3 u_1}{\partial x_1 \partial t_1^2} = 0. \quad \text{(A 4)}$$

Equation (A 3) can be rewritten

$$2\frac{\partial u_2}{\partial t_1} + 2c_0 \frac{\partial u_2}{\partial x_1} + 3u_1 \frac{\partial u_1}{\partial x_1} + bc_0 u_1 \frac{\partial u_1}{\partial t_1} + \frac{(\psi_B - \psi_A)}{c_0} \frac{\partial^3 u_1}{\partial x_1 \partial t_1^2} = \left(\frac{\partial}{\partial t_1} + c_0 \frac{\partial}{\partial x_1}\right)(u_2 - 2c_2).$$

Operate on this equation with

$$\left(\frac{\partial}{\partial t_1} - c_0 \frac{\partial}{\partial x_1}\right)$$

and then substitute for

$$\left(\frac{\partial}{\partial t_1} - c_0 \frac{\partial}{\partial x_1}\right)(u_2 - 2c_2)$$

from (A 4) to find

$$\left(\frac{\partial}{\partial t_1} - c_0 \frac{\partial}{\partial x_1}\right)\left[2\frac{\partial u_2}{\partial t_1} + 2c_0 \frac{\partial u_2}{\partial x_1} + 3u_1 \frac{\partial u_1}{\partial x_1} + bc_0 u_1 \frac{\partial u_1}{\partial t_1} + \frac{(\psi_B - \psi_A)}{c_0} \frac{\partial^3 u_1}{\partial x_1 \partial t_1^2}\right]$$
$$= \left(\frac{\partial}{\partial t_1} + c_0 \frac{\partial}{\partial x_1}\right)\left[bc_0 u_1 \frac{\partial u_1}{\partial t_1} + \frac{(\psi_B - \psi_A)}{c_0} \frac{\partial^3 u_1}{\partial x_1 \partial t_1^2}\right]. \quad \text{(A 5)}$$

But

$$\frac{\partial u_1}{\partial t_1} = -c_0 \frac{\partial u_1}{\partial x_1}, \quad \text{(A 6)}$$

so that the right-hand side of (A 5) is zero and the left-hand side may be integrated to give

$$2\frac{\partial u_2}{\partial t_1} + 2c \frac{\partial u_2}{\partial x_1} + (3 - bc_0^2)\, u_1 \frac{\partial u_1}{\partial x_1} - (\psi_B - \psi_A) \frac{\partial^3 u_1}{\partial x_1^2 \partial t_1} = f(x_1 + c_0 t_1), \quad \text{(A 7)}$$

where further use has been made of (A 6). If the waves are travelling into still water $f(x_1+c_0t_1)$ is zero, and in other cases it is zero if the initial conditions are appropriate.

Equation (22) is now obtained by adding ϵ times (A 6) to ϵ^2 times (A 7) and putting $u' = \epsilon u_1 + \epsilon^2 u_2$ so that

$$2\frac{\partial u'}{\partial t} + 2c_0\frac{\partial u'}{\partial x} + (3 - bc_0^2)u'\frac{\partial u'}{\partial x} = (\psi_B - \psi_A)\frac{\partial^3 u'}{\partial x^2\,\partial t}.$$

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